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ABSTRACT

The Generalized Minimum Spanning Tree Problem (GMSTP) is a variant of the classical Minimum Spanning Tree Problem, in which the vertices are partitionned into clusters and the tree has to include exactly one vertex from each cluster. A new integer linear formulation is presented for the problem and the facial structure of the corresponding polytope is analyzed. These results are extended to more general polytopes.

Keywords: Minimum Spanning Tree Problem, Network Design, Polyhedral Analysis, Multiple Choice.

RÉSUMÉ

Le problème de l'arbre de recouvrement minimal généralisé est une variante du problème classique de l'arbre de recouvrement minimal dans lequel l'ensemble des sommets est partitionné en sous-ensembles et où l'arbre doit être incident à exactement un sommet par sous-ensemble. On formule ce problème comme un programme linéaire en nombres entiers et on en analyse la structure faciale ainsi que le polytope correspondant. Ces résultats sont étendus à des polytopes plus généraux.

Mots-clés: Arbre de recouvrement minimal, conception de réseaux, analyse polyédrale, choix multiple.

1. Introduction

The *Generalized Minimum Spanning Tree Problem* (GMSTP) is defined on an undirected graph $G = (V, E)$, where V is a vertex set partitioned into $|K|$ *clusters* V_k, $k \in K = \{1, \cdots, |K|\}$, and E is the edge set. With each edge $e \in E$ is associated a *cost* $c_e \in \mathbf{R}$ and with each node $i \in V$ is associated a *cost* $d_i \in \mathbf{R}$. The GMSTP consists of determining a minimum cost tree including exactly one vertex from each cluster. Applications of this problem are encountered in telecommunications where local networks must be interconnected by a tree containing a gateway from each cluster (see Myung, Lee, Tcha [11]). These authors show that the GMSTP is NP-hard by reduction from NODE COVER, and that unless $P \neq NP$, no polynomial-time heuristic with a bounded worst-case performance ratio can exist for this problem. They also provide several integer linear programming formulations of the problem, as well as a heuristic and a branch-and-bound algorithm.
Our aim is to investigate the polyhedral structure of the GMSTP and to relate it to that of two of its relaxations. Specifically, we consider polytopes related to the following structures:

1. the *Generalized Subgraph* (GS) containing at most one vertex from each cluster;
2. *the Generalized Acyclic Subgraph* (GAS);
3. *the Generalized Spanning Tree* (GST), i.e. the GAS spanning all clusters.

The GS subsumes several network design structures including those of the *Generalized Traveling Salesman Problem* (Fischetti, Salazar and Toth [5,6]), of several *Location-Routing Problems* [9], of the *Cycle Problem* (Bauer [1]), and of the *Steiner Tree Problem* (Chopra and Rao [2,3], Goemans and Myung [7], Goemans [8], Salazar [16]). The GAS subsumes the *Generalized Shortest Path Problem* structure (Li, Tsao and Ulular [15]). Hence, any valid inequality for the GS and the GAS polytopes applies to all of their restrictions.

The remainder of this article is organized as follows. In Section 2, we introduce some notation and provide integer linear formulations for optimization problems associated with the GS, GAS and GST structures. In Section 3, we provide results on polyhedra defined in the space of edge and vertex variables. In Section 4, polytopes defined in the space of edge variables only are analyzed. A summary and conclusion are provided in Section 5.

2. Notation and formulation

In what follows, it is assumed that $|K| \geq 5$ and that $(V_1, \cdots, V_{|K|}, E)$ is a complete $|K|$-partite graph . We define the binary variables

$$x_e = x_{ij} = \begin{cases} 1 & \text{if } e = \{i,j\} \in E \text{ is selected in a solution,} \\ 0 & \text{otherwise,} \end{cases}$$
$$\text{and} \quad y_i = \begin{cases} 1 & \text{if } i \in V \text{ is selected in a solution,} \\ 0 & \text{otherwise.} \end{cases}$$

The *incidence vector* of a subgraph $H = (V(H), E(H))$ of G is defined as (x, y), with binary components $x_e = 1$ if and only if $e \in E(H)$, and $y_i = 1$ if and only if $i \in V(H)$. For simplicity, when there is no ambiguity, a subgraph H may be represented by its edge set $E(H)$. For $F \subseteq E$, define $x(F) = \sum_{e \in F} x_e$; similarly, for $S \subseteq V$, define $y(S) = \sum_{i \in S} y_i$. Also for $S \subseteq V$, let

$$\begin{aligned} \bar{S} &= V \backslash S, \\ \delta(S) &= \{e = \{i,j\} \in E : i \in S, j \in \bar{S}\}, \\ E(S) &= \{e = \{i,j\} \in E : i, j \in S\}, \\ \mu(S) &= |\{k : V_k \subseteq S\}|, \\ \eta(S) &= |\{k : V_k \cap S \neq \emptyset\}|. \end{aligned}$$

For $S, T \subseteq V$, define $\delta(S,T) = \{e = \{i,j\} \in E : i \in S, j \in T\}$. For $v \in V$, denote by $V_{h(v)}$ the cluster containing v. Also define $W = \{v \in V : |V_{h(v)}| = 1\}$. Let u_e^E be the $|E|$-dimensional unit vector whose coordinate corresponding to edge e is equal to 1. Similarly, u_i^V denotes the $|V|$-dimensional unit vector whose coordinate corresponding to vertex i is equal to 1. The GMSTP can be formulated as an integer linear program as follows.

$$\min \sum_{e \in E} c_e x_e + \sum_{i \in V} d_i y_i$$

s.t.

$$\begin{aligned} y(V_k) &\leq 1 && k \in K, && (1) \\ x(E(S)) &\leq y(S \backslash V_k) && \emptyset \subset S \subset V, k \in K, && (2) \\ x(E) &= |K| - 1, && && (3) \\ x_e &\in \{0,1\} && e \in E, && (4) \\ y_i &\in \{0,1\} && i \in V. && (5) \end{aligned}$$

Denote by $(2, k, S)$ constraint (2) relative to a given subset S and a given cluster k. Also define $\mathcal{P}_{123} = \text{conv}\,\{(x,y) \in \{0,1\}^{|E|\times|V|} \text{ s.t.}(1),(2),(3)\}$, the polytope associated with the GST, by $\mathcal{P}_{12} = \text{conv}\,\{(x,y) \in \{0,1\}^{|E|\times|V|} \text{ s.t.}(1),(2)\}$, the polytope associated with the GAS, and by $\mathcal{P}_1 = \text{conv}\,\{(x,y) \in \{0,1\}^{|E|\times|V|} \text{ s.t. } (1), x_{ij} \leq y_i, x_{ij} \leq y_j\}$, the polytope associated with the GS. When $|V_k| = 1$ for all $k \in K$, $\mathcal{P}_{123}$ coincides with $\mathcal{P}_{ST}$, the convex hull of the incidence vectors of all the spanning trees of G (see Chopra [4]). When all vertex costs are zero, a minimum cost GST solution can be obtained by solving a minimum cost GAS in which a sufficiently large constant is substracted from each edge cost.

Note that $\mathcal{P}_{123} \subset \mathcal{P}_{12} \subset \mathcal{P}_1$ and $\mathcal{P}_{123}$ is a face of $\mathcal{P}_{12}$. Further, in $\mathcal{P}_{123}$, the constraints (1) can be replaced by $y(V_k) = 1, \forall k \in K$. Indeed, if $y(V_j) < 1$ for a certain $j \in K$, constraint $(2, k, V)$ with $k \neq j$ yields $x(E) \leq \sum_{p \in K \backslash \{k\}} y(V_p) < |K| - 1$ which contradicts constraint (3). Constraints (5) can be replaced by $y \geq 0$ for $\mathcal{P}_{123}$. If there existed a vertex v with a fractional y_v, one would have $y_i < 1$ for all $i \in V_{h(v)}$. Then, constraints $x_{ij} \leq y_i$ for $i \in V_{h(v)}$ would imply $x(\delta(V_{h(v)})) = 0$. Since constraint (3) can be rewritten as $x(E(V \backslash V_{h(v)})) + x(\delta(V_{h(v)})) = |K| - 1$, it would contradict constraint $(2, k, V \backslash V_{h(v)})$ with $k \neq h(v)$. For $\mathcal{P}_{123}$ and $\mathcal{P}_{12}$, constraint $(2, k, S)$ is dominated by constraint $(2, k, S \cup V_k)$ whenever $\mu(S) = 0$.

3. Facet defining inequalities for $\mathcal{P}_{123}$, $\mathcal{P}_{12}$ and $\mathcal{P}_1$

We now derive facet defining inequalities for polytopes $\mathcal{P}_{123}, \mathcal{P}_{12}$ and $\mathcal{P}_1$. We first determine the dimension of these polytopes. In what follows we use some definitions from Fischetti, Salazar and Toth [5].

Definition 1. *Given a valid inequality $\alpha x \leq \beta y + \gamma$ for $\mathcal{P}_{123}$, let $\mathcal{H}(\alpha,\beta,\gamma) = \mathcal{P}_{123} \cap \{(x,y) \in \mathbb{R}^{|E|\times|V|}; \alpha x = \beta y + \gamma\}$ denote the face of $\mathcal{P}_{123}$ induced by $\alpha x \leq \beta y + \gamma$. Let $v \in V\backslash W$ be an arbitrary but fixed vertex and let $\mathcal{P}^v_{123}$ denote the GST polytope associated with the subgraph of G induced by $V\backslash\{v\}$. The v-restriction of $\alpha x \leq \beta y + \gamma$ is the inequality obtained from $\alpha x \leq \beta y + \gamma$ by dropping the variables y_v and x_e for all $e \in \delta(v)$. Similarly, let $i, j \in V\backslash W$ be arbitrary but fixed vertices, and let $\mathcal{P}^{ij}_{123}$ denote the GST polytope associated with the subgraph of G induced by $V\backslash\{i,j\}$. The ij-restriction of $\alpha x \leq \beta y + \gamma$ is the inequality obtained from $\alpha x \leq \beta y + \gamma$ by dropping the variables y_i, y_j and x_e for all $e \in \delta(i) \cup \delta(j)$.*

Lemma 1. *For every valid inequality $\alpha x \leq \beta y + \gamma$ for $\mathcal{P}_{123}$ and every vertex $v \in V\backslash W$, the dimension of $\mathcal{H}(\alpha,\beta,\gamma)$ is greater than or equal to the dimension of the face of $\mathcal{P}^v_{123}$ induced by its v-restriction, plus the rank of the matrix formed by the coordinates (x,y), restricted to x_e for $e \in \delta(v)$ and y_v, of the extreme points of $\mathcal{H}(\alpha,\beta,\gamma)$ with $y_v = 1$.*

Proof. Let X be the matrix in which every row is an extreme point of $\mathcal{H}(\alpha,\beta,\gamma)$. Since $\mathcal{H}(\alpha,\beta,\gamma)$ is contained in a hyperplane not passing through the origin (e.g. induced by constraint (3)), the dimension of $\mathcal{H}(\alpha,\beta,\gamma)$ coincides with the rank of X, minus 1. Since $v \notin W$, X can be partitionned into $X = \begin{pmatrix} X_{11} & 0 & 0 \\ X_{21} & X_{22} & 1 \end{pmatrix}$, where the last column corresponds to variable y_v, and the columns of X_{22} correspond to variables $x_e \in \delta(v)$. Since rank $X \geq$ rank X_{11} + rank $(X_{22} \quad 1)$, and the dimension of the face of $\mathcal{P}^v_{123}$ induced by the v-restriction of $\alpha x \leq \beta y + \gamma$ is equal to the rank of X_{11}, minus 1, then dim($\mathcal{H}(\alpha,\beta,\gamma)$) $\geq$ dim (face of $\mathcal{P}^v_{123}$ induced by the v-restriction) + rank $(X_{22} \quad 1)$. □

Proposition 1. *The polytopes $\mathcal{P}_{12}, \mathcal{P}_1$ are full-dimensional. Further, dim($\mathcal{P}_{123}$) $= |E| + |V| - |K| - 1$.*

Proof. It is easy to show that the canonical basis in the (x,y)-space is in $\mathcal{P}_{12}$, and that the origin belongs to $\mathcal{P}_{12}$. Clearly, $\dim(\mathcal{P}_{123}) \leq |E| + |V| - |K| - 1$ since equations $y(V_k) = 1, k \in K$ and $x(E) = |K| - 1$ are valid for $\mathcal{P}_{123}$ and are linearly independent. We use induction on $\rho = |V| - |K|$ to prove the reverse inequality. When $\rho = 0$, then $\mathcal{P}_{123} = \mathcal{P}_{ST}$ and the claim is true (see Chopra [4]). Assume now that the claim holds for $\rho = \bar{\rho}$, and consider any GST polytope with $\rho = \bar{\rho} + 1$. Then, there exists a vertex $v \in V\backslash W$. This induction hypothesis with Lemma 1 for the trivial valid inequality $0 \leq 0$ leads to $\dim(\mathcal{P}_{123}) \geq |E\backslash\delta(v)| + |V\backslash\{v\}| - |K| - 1 +$ rank $(X_{22} \quad 1)$. Thus, if we show that rank $(X_{22} \quad 1) = |\delta(v)| + 1$, we are done. This is equivalent to exhibiting $|\delta(v)| + 1$ incidence vectors of GST containing vertex v such that their restriction to x_e for all $e \in \delta(v)$ and y_v are linearly independent.
Assume for simplicity that $h(v) = 1$ and let v_k be a vertex arbitrarily chosen in V_k for each $k \in K\backslash\{1\}$. For every vertex $p \in V\backslash V_1$, we consider the following GST containing vertex v: $\{(v,p)\} \cup \{(p,v_k) : k \in K\backslash\{h(p),1\}\}$. These trees show that rank $(X_{22}) \geq \text{rank}(I_{|\delta(v)|\times|\delta(v)|}) = |\delta(v)|$. Since $|K| \geq 5$, there exist at least two clusters different from V_1. The restriction to x_e for all $e \in \delta(v)$ and y_v of the incidence vector of the GST $\{(v,v_2),(v,v_3)\} \cup \{(v_2,v_k) : k \in K\backslash\{1,2,3\}\}$ is obviously linearly independent from the previous $|\delta(v)|$ vectors. □

Lemma 2. *Let $\alpha x \leq \beta y + \gamma$ be a valid inequality $\alpha x \leq \beta y + \gamma$ inducing a proper face of $\mathcal{P}_{123}$. If there exists a vertex $v \in V\backslash W$ such that the two following conditions hold:*

(i) the v-restriction of this inequality is facet defining for $\mathcal{P}^v_{123}$, and
(ii) there exist $|\delta(v)| + 1$ incidence vector of GST of the graph G containing vertex v, satisfying $\alpha x = \beta y + \gamma$, and such that their restriction to x_e for all $e \in \delta(v)$ and y_v are linearly independent,

then $\alpha x \leq \beta y + \gamma$ induces a facet of $\mathcal{P}_{123}$.

Proof. Since $\mathcal{H}(\alpha,\beta,\gamma)$ is a proper face of $\mathcal{P}_{123}$, then $\dim(\mathcal{H}(\alpha,\beta,\gamma)) \leq |E| + |V| - |K| - 2$. Hypothesis (i) and Proposition 1 imply that $\dim(v\text{-restriction}) = |E\backslash\delta(v)| + |V\backslash v| - |K| - 2 = |E| + |V| - |K| - |\delta(v)| - 3$. Using the notation of Lemma 1, hypothesis (ii) is equivalent to $\text{rank}(X_{22}1) \geq |\delta(v)| + 1$. Lemma 1 implies $\dim(\mathcal{H}(\alpha,\beta,\gamma)) \geq |E| + |V| - |K| - 2$. Therefore $\dim(\mathcal{H}(\alpha,\beta,\gamma)) = |E| + |V| - |K| - 2$ and $\alpha x \leq \beta y + \gamma$ defines a facet of $\mathcal{P}_{123}$. □

We are now ready to study several families of facet-inducing valid inequalities for $\mathcal{P}_{123}, \mathcal{P}_{12}$ and $\mathcal{P}_1$. We first study conditions under which trivial inequalities are facet defining.

Proposition 2. *1. The inequality $x_e \geq 0$, $e \in E$, defines a facet of $\mathcal{P}_{123}, \mathcal{P}_{12}$ and $\mathcal{P}_1$.*
2. The inequality $y_i \geq 0$, $i \in V$ does not define a facet of $\mathcal{P}_{123}, \mathcal{P}_{12}$ and $\mathcal{P}_1$.

3. The inequality $y(V_k) \leq 1, k \in K$ does not define a facet of $\mathcal{P}_{123}$, but defines a facet of $\mathcal{P}_{12}$ and $\mathcal{P}_1$.

Proof. 1. Without loss of generality, let $e = (1,2), h(1) = 1, h(2) = 2$. Let v_k be a vertex arbitrarily chosen in each V_k for $k \in K$. We first prove by induction on $\rho = |V| - |K|$ that $x_e \geq 0$ induces a facet of $\mathcal{P}_{123}$. When $\rho = 0$, then $\mathcal{P}_{123} = \mathcal{P}_{ST}$ and therefore $x_e \geq 0$ is facet inducing (Chopra [4]). Assume now that the claim holds for $\rho = \bar{\rho}$ and consider any GST instance with $\rho = \bar{\rho} + 1$. There exists a vertex $v \notin \{1,2\} \cup W$. By Lemma 2 and the induction hypothesis, if we exhibit $|\delta(v)| + 1$ GST containing vertex v and satisfying the assumptions of Lemma 2, we are done. Two cases can occur.

(a) If $v \in V_1 \cup V_2$, note that $y_v = 1$ implies that $x_e = 0$. Assume that $v \in V_1$. In this case, the $|\delta(v)|$ vectors correspond to the GST $\{(v,p)\} \cup \{(p,v_k) : k \in K \backslash \{1, h(p)\}\}, p \in V \backslash V_1$. Since $|K| \geq 3$ there exist a vertex p such that $h(p) \neq 1,2$, and the last needed vector can be $\{(v,p),(v,v_2)\} \cup \{(p,v_k) : k \in K \backslash \{1,2,h(p)\}\}$. Obviously, the restricted vectors are linearly independent.

(b) If $v \notin V_1 \cup V_2$, without loss of generality, assume that $h(v) = 3$. For every $p \in V \backslash (V_1 \cup V_2 \cup V_3)$, the GST $\{(v,p),(p,1),(p,2)\} \cup \{(p,v_k) : k \in K \backslash \{1,2,3,h(p)\}\}$ does not contain the edge $e = (1,2)$. Since we have assumed that $|K| \geq 5$, for $p \in V_1$ (resp. $p \in V_2$), there exists a vertex $q \in V_4$ such that the GST $\{(v,p),(p,q)\} \cup \{(q,v_k) : k = 5, \cdots, |K|\} \cup \{(q,2)\}$ (resp. $\cup\{(q,1)\}$) does not contain the edge $e = (1,2)$. Their incidence vectors are linearly independent when their coordinates are restricted to $x_e, e \in \delta(v)$ and y_v. The last incidence vector of the GST $\{(v,1),(v,2)\} \cup \{(2,v_k) : k = 4, \cdots, |K|\}$ completes the requested set.

The inequality $x_e \geq 0$ is facet defining for $\mathcal{P}_{12}$ and $\mathcal{P}_1$. Indeed, since the origin belongs to the face, it suffices to exhibit $|V| + |E| - 1$ linearly independent vectors in $\mathcal{P}_{12}$ satisfying $x_e = 0$. Now define $|V|$ vectors (x,y) with $x = 0$ and $y = u_i^V$ for $i \in V$, and $|E| - 1$ vectors (x,y) with $x = u_f^E (f = \{i,j\} \in E \backslash \{e\})$ and $y = u_i^V + u_j^V$. These vectors are clearly linearly independent.

2. The inequality $y_i \geq 0$ does not define a facet for any of the three polytopes. Indeed, $\mathcal{P} \cap \{(x,y) : y_i = 0\} \subset \mathcal{P} \cap \{(x,y) : x(\delta(i) \cap \delta(V_j)) = 0\} \subset \mathcal{P}$, where $\mathcal{P} = \mathcal{P}_{123}, \mathcal{P}_{12}$ or $\mathcal{P}_1, j \in K, j \neq h(i)$.

3. Constraints (1) do not define facets of $\mathcal{P}_{123}$ since they are tight for all points of $\mathcal{P}_{123}$. To show that they induce facets of $\mathcal{P}_{12}$, and $\mathcal{P}_1$, it suffices to exhibit $|V| + |E|$ linearly independent vectors of $\mathcal{P}_{12}$ satisfying $y(V_k) = 1$. Let v be one vertex of V_k. Now consider the $|V_k|$ vectors (x,y) such that $x = 0$ and $y = u_i^V, i \in V_k$, the $|V \backslash V_k|$ vectors (x,y) such that $x = 0$ and $y = u_v^V + u_j^V, j \in V \backslash V_k$, and the $|E|$ vectors (x,y) such that $x = u_e^E, e = \{i,j\} \in E$ and $y = u_i^V + u_j^V + u_v^V$ if $\{i,j\} \cap V_k = \emptyset$, and $y = u_i^V + u_j^V$ otherwise. The vectors are linearly independent. □

We now revert our attention to the cycle elimination constraints $(2,k,S), k \in K, V_k \subset S \subset V$.

Proposition 3. *Let $k \in K, \mu(S) \neq 0$ and $V_k \subset S \subset V, |S| \geq 2$. Then inequality $(2,k,S)$ is facet defining for $\mathcal{P}_{123}$ if and only if any of the following four conditions holds:*

(i) $\eta(S) \geq 3$ and $\eta(V \backslash S) \geq 2$,
(ii) $S \subseteq W, |S| = 2$,
(iii) $S = \{v\} \cup V_k$ with $v \notin W \cup V_k$,
(iv) $S = V \backslash \{v\}, v \in V$.

Proof. Let v_k be a vertex arbitrarily chosen in each V_k for $k = 1, \cdots, |K|$. Using constraint (3), constraint $(2,k,S)$ can be rewritten for $\mathcal{P}_{123}$ as
$x(E(S)) \leq y(S) - 1$.

1. The *sufficient* condition is proven in four steps.

1.1. We first show that if (i) is satisfied, then constraint $(2,k,S)$ is facet defining for $\mathcal{P}_{123}$. Since $x(E(S)) \leq |S| - 1$ defines a facet of $\mathcal{P}_{ST}$ when G is complete, using Lemma 2, it is sufficient to show that for any given $v \notin W$, $\text{rank}(X_{22} \quad 1) \geq |\delta(v)| + 1$. To this end, we exhibit $|\delta(v)| + 1$ GST containing vertex v such that their restriction to S remains a tree and the restriction of their incidence vectors to y_v and x_e for all $e \in \delta(v)$ are linearly independent. Without loss of generality, assume that $h(v) = 1$. There are two cases:

Case 1. Vertex $v \notin S \cup W$. Let $q \in S \backslash V_1$ be such that $V_{h(q)} \subset S$. For each $p \notin S \cup V_1$, construct the following GST $\{(v,p),(p,q)\} \cup \{(p,v_k) : v_k \notin S, k \neq h(p), h(q), 1\} \cup \{(q,v_k) : v_k \in S, k \neq h(q), h(p), 1\}$. For each $p \in S \backslash V_1$, construct the following GST:

- $\{(v,p)\} \cup \{(p,v_k) : v_k \in S, k \neq h(p), 1\}$ if $S \cup V_1 \cup V_{h(p)} = V$,
- $\{(v,p),(p,q)\} \cup \{(p,v_k) : v_k \in S, k \neq h(p), 1\} \cup \{(q,v_k) : v_k \notin S, k \neq h(q), h(p), 1\}$ if there exists $q \in V \backslash (S \cup V_1 \cup V_{h(p)})$.

We need one more vector . Since $\eta(V\backslash S) \geq 2$, there exists a vertex $q \in V\backslash(S \cup V_1)$. Let $p \in S\backslash V_1$ be such that $V_{h(p)} \subset S$. One GST that fits our requirement is $\{(v,q),(v,p)\} \cup \{(q,v_k) : v_k \notin S, k \neq h(q), h(p), 1\} \cup \{(p,v_k) : v_k \in S, k \neq h(p), h(q), 1\}$. This implies that for $v \notin S \cup W$, the conditions of the Lemma 2 are satisfied.

Case 2. Vertex $v \in S\backslash W$.

For each $p \notin S \cup V_1$, construct the following GST:

- $\{(v,p)\} \cup \{(p,v_k) : v_k \notin S, k \neq h(p), 1\}$ if $S\backslash(V_1 \cup V_{h(p)}) = \emptyset$,
- $\{(v,p),(v,q)\} \cup \{(p,v_k) : v_k \notin S, k \neq h(p), h(q), 1\} \cup \{(q,v_k) : v_k \in S, k \neq h(q), i \neq h(p)\}$ if there exists $q \in S\backslash(V_1 \cup V_{h(p)})$.

For each $p \in S \cup V_1$, construct the following GST:

- $\{(v,p)\} \cup \{(p,v_k) : v_k \in S, k \neq h(p), 1\}$ if $V\backslash(S \cup V_1 \cup V_{h(p)}) = \emptyset$,
- $\{(v,p),(p,q)\} \cup \{(p,v_k) : v_k \in S, k \neq h(p)h(q), 1\} \cup \{(q,v_k) : v_k \notin S, k \neq h(p), h(q), 1\}$ if there exists $q \in V\backslash(S \cup V_1 \cup V_{h(p)})$.

We need one more vector. Since $\eta(S) \geq 3$, there exist two vertices $p,q \in S$ such that $h(p) \neq h(q), h(p), h(q) \neq 1$. One GST that fits our requirement is:

- $\{(v,p),(v,q)\} \cup \{(p,v_k) : v_k \in S, k \neq h(p), h(q), 1\}$ if $V\backslash(S \cup V_1 \cup V_{h(p)} \cup V_{h(q)}) = \emptyset$,
- $\{(v,p),(v,q),(p,r)\} \cup \{(p,v_k) : v_k \in S, k \neq h(p), h(q), 1\} \cup \{(r,v_k) : v_k \notin S, k \neq h(r), h(p), h(q), 1\}$ if there exists $r \in V\backslash(S \cup V_1 \cup V_{h(p)} \cup V_{h(q)})$.

This implies that the constraints $x(E(S)) \leq y(S) - 1$ with $\mu(S) \neq 0$ are facet defining for $\mathcal{P}_{123}$ when $\eta(S) \geq 3$ and $\eta(V\backslash S) \geq 2$.

1.2. Assume that (ii) is satisfied. Since $x_e \leq 1$ is facet defining for $\mathcal{P}_{ST}$, it is sufficient to exhibit $|\delta(v)| + 1$ GST satisfying the conditions of Lemma 2. Let $e = (1,2)$, $v \notin W$, $h(1) = 1, h(2) = 2, h(v) = 3$. The $|\delta(v)|$ vectors correspond to the GST $\{(1,2),(v,p)\} \cup \{(p,v_k) : k = 4, \cdots, |K|\}$ for $p = 1,2$, and $\{(1,2),(v,p)\} \cup \{(p,v_k) : k = 2, \cdots, |K|, k \neq 3, h(p)\}$ for p such that $h(p) \neq 1,2,3$. Since $|K| \geq 5$, there exists p such that $h(p) \neq 1,2,3$, and the last required vector can be, $\{(1,2),(v,1),(v,p)\} \cup \{(p,v_k) : k = 4, \cdots, |K|, k \neq h(p)\}$ and we are done.

1.3. Assume that (iii) is satisfied. In this case, $(2,k,S)$ can be rewritten as $x(\delta(v,V_k)) \leq y_v$. The v-restriction of this constraint is the trivial inequality $0 \leq 0$, so according to Lemma 1 it is sufficient to exhibit $|\delta(v)|$ GST containing vertex v and satisfying the assumptions of Lemma 1. Let vertex $q \in V_k$. The following trees $\{(v,p)\} \cup \{(p,v_l) : l \neq k, h(v)\}, p \in V_k$, and $\{(v,p),(p,q)\} \cup \{(p,v_l) : l \neq k, h(v), h(p)\}, p \notin V_k \cup V_{h(v)}$, satisfy the requirements.

1.4. Assume that (iv) is satisfied. In this case, constraint $(2,k,S)$ can be rewritten $y_v \leq x(\delta(v))$. We use induction on $\rho = |V| - |K|$. When $\rho = 0$, $\mathcal{P}_{123} = \mathcal{P}_{ST}$ and the claim is true. Assume now that the claim holds for $\rho = \bar{\rho}$, and consider any GST polytope with $\rho = \bar{\rho} + 1$. Then, there exists a vertex $u \in V\backslash W$. Two cases have to be considered: $u \in S \cap V_{h(v)}$ or $u \in S\backslash V_{h(v)}$.

Case 1. If $u \in S \cap V_{h(v)}$, it suffices to exhibit $|\delta(u)| + 1$ generalized spanning trees satisfying the conditions of Lemma 2. They have to satisfy $x(\delta(v)) = y_v$ and $y_u = 1$. But if $y_u = 1$ the inequality becomes trivial $(0 = 0)$.

Case 2. If $u \in S\backslash V_1$, it suffices to exhibit $|\delta(u)| + 1$ generalized spanning trees satisfying the conditions of Lemma 2. The following generalized spanning trees satisfy our requirement. Since $|K| \geq 3$, there exists $q \in S$ with $h(q) \neq 1, h(u)$, a first tree can be: $\{(u,v),(u,q)\} \cup \{(q,v_k) : k \neq h(q), h(u), v\}$. The others $|\delta(u)| - 1$ can be for example:

- $\{(u,p),(p,v)\} \cup \{(p,v_k) : k \neq h(p), h(v), h(u)\}$ if $p \in S\backslash(V_{h(u)} \cup V_{h(v)})$,
- $\{(u,p)\} \cup \{(p,v_k) : k \neq h(v), h(u)\}$ if $p \in S \cap V_{h(v)}$.

Since $|K| \geq 4$, there exist vertices $q,p \in S$ such that $h(q) \neq h(p), h(p), h(q) \neq h(v), h(u)$ and this leads to the last vector: $\{(u,p),(u,q),(p,v)\} \cup \{(p,v_k) : k \neq h(p), h(q), h(v), h(u)\}$.

2. We now prove the ***necessary*** condition. If none of the four conditions (i), (ii), (iii), (iv) is satisfied, then one of the following two cases must hold.

Case 1. Either $S = V_k \cup \bar{S}$ with $\bar{S} \subseteq V_l, l \neq k$ and $|\bar{S}| \geq 2$. Let u be a vertex in $\bar{S}$. We observe that the vectors of $\mathcal{P}_{123}$ on the face induced by constraint $(2,k,S)$ belong to the hyperplane defined by $x(\delta(u,V_k)) = y_u$.

Case 2. Or $\bar{S} \subseteq V_l, l \neq k$ with $|\bar{S}| \geq 2$, let u be a vertex in $\bar{S}$. In this case $(2,k,S)$ can be rewritten as $y(\bar{S}) \leq x(\delta(S))$. We then note that the vectors of $\mathcal{P}_{123}$ on the face induced by constraint $(2,k,S)$ are included in the hyperplane defined by $x(\delta(u)) = y_u$. □

Proposition 4. *Let $k \in K, \mu(S) \neq 0$ and $V_k \subset S \subset V, |S| \geq 2$. Then the inequality $(2,k,S)$ is facet defining for $\mathcal{P}_{12}$ if and only if any of the following two conditions holds:*

(i) $\eta(S) \geq 3$,
(ii) $S = \{v\} \cup V_k$ *with* $v \notin V_k$.

Proof. We show that constraint $(2, k, S)$ with $\mu(S) \neq 0$ and $V_k \subset S \subset V$ defines a facet of $\mathcal{P}_{12}$ if $\eta(S) \geq 3$ or $S = V_k \cup \{v\}, v \notin V_k$. We use the indirect method that consists of showing that any other inequality $\sum_{e \in E} \alpha_e x_e + \sum_{i \in V} \beta_i y_i \leq \gamma$ such that $\{(x,y) \in \mathcal{P}_{123} : x(E(S)) = y(S \backslash V_k)\} \subseteq \{(x,y) \in \mathcal{P}_{123} : \alpha x + \beta y = \gamma\}$ is a non-negative multiple of $x(E(S)) \leq y(S \backslash V_k)$. Since the origin is in the face, this implies $\gamma = 0$.

We first prove that all coefficients α_e associated with the edges $e \in E(V \backslash (S \backslash V_k))$ and all coefficients β_i associated with vertex $i \in (V \backslash S) \cup V_k$ are equal to zero. For this purpose consider the vectors (x, y) such that $x = u_e^E (e = \{i, j\} \in E(V \backslash (S \backslash V_k)))$, $y = u_i^V + u_j^V$ and the vectors defined by $x = 0, y = u_i (i \in V \backslash (S \backslash V_k))$. These vectors all belong to the face induced by $(2, k, S)$ which implies $\alpha_e = 0, e \in E(V \backslash (S \backslash V_k)), \beta_i = 0, i \in V \backslash (S \backslash V_k)$.

For the second step, we consider the vectors (x, y) defined by $x = u_e^E (e = \{i, m\}, i \in S \backslash V_k, m \in V_k), y = u_i^V + u_m^V$. These vectors all belong to the face, it implies that $\alpha_e = -\beta_i, e \in \delta(i) \cap \delta(V_k), i \in S \backslash V_k$.

In the third step, we observe that the vectors defined by $x = u_e^E + u_f^E, e = \{i, m\}, i \in S \backslash V_k, m \in V_k, f = \{i, j\}, j \in V \backslash S, h(j) \neq h(i), y = u_i^V + u_j^V + u_m^V$ belong to the face, which implies that $\alpha_e = 0, e \in \delta(S) \cap \delta(S \backslash V_k)$. This shows that $(2, k, S)$ defines a facet of $\mathcal{P}_{12}$ when $S = V_k \cup \{v\}, v \notin V_k$. To conclude, if $\eta(S) \geq 3$, consider the vectors of the face defined by $x = u_e^E + u_f^E, e = \{i, m\}, i \in S \backslash V_k, m \in V_k, f = \{i, j\}, j \in S \backslash V_k, h(j) \neq h(i), y = u_i^V + u_j^V + u_m^V$, they imply that $\alpha_e = -\beta_j, e = \{i, j\} \in E(S \backslash V_k)$. It follows that constraint $(2, k, S)$ defines a facet of $\mathcal{P}_{12}$ when $\eta(S) \geq 3$.
Now, if $\eta(S) = 2$ with $|S \backslash V_k| \geq 2$, then the vectors in the face induced by constraint $(2, k, S)$ satisfy the non valid equality $x(\delta(i) \cap \delta(V_k)) = y_i, i \in S \backslash V_k$.

□

Proposition 5. *Let* $k \in K$, $\mu(S) \neq 0$ *and* $V_k \subset S \subset V, |S| \geq 2$. *Then the inequality* $(2, k, S)$ *is facet defining for* $\mathcal{P}_1$ *if and only if* $S = \{v\} \cup V_k$ *with* $v \notin V_k$.

Proof. The constraint $(2, k, S)$ is valid for $\mathcal{P}_1$ if and only if $\eta(S) = 2$ and the argument used for $\mathcal{P}_{12}$ can be applied to prove that constraint $(2, k, S)$ is facet defining if and only if $|S \backslash V_k| = 1$. □

To conclude this section we present results on polytopes in the (x, y)-space.

Proposition 6. *(Hammock constraints). Let* $r, s \in V$ *such that* $h(r) \neq h(s)$, $|V_{h(r)}| \geq 2, |V_{h(s)}| \geq 2$. *Given an index set* $\tilde{K} \subseteq K \backslash \{h(r), h(s)\}$ *with* $|\tilde{K}| \geq 2$, *the inequality*

$$(|\tilde{K}| - 1)x_{rs} + y_r + y_s + x(\bigcup_{k \in \tilde{K}} \delta(V_k, (V_{h(r)} \cup V_{h(s)}) \backslash \{r, s\})) + x(\delta(V_{h(r)} \backslash \{r\}, V_{h(s)} \backslash \{s\})) \leq |\tilde{K}| + 1 \qquad (6)$$

is valid for $\mathcal{P}_{12}$ *(and thus for* $\mathcal{P}_{123}$*). However, this constraint is not valid for* $\mathcal{P}_1$.

Proof. To simplify the notation, we can set $r = 1, s = 2, h(r) = 1$ and $h(s) = 2$ without loss of generality. Thus (6) can be rewritten as

$$(|\tilde{K}| - 1)x_{12} + y_1 + y_2 + x(\bigcup_{k \in \tilde{K}} \delta(V_k, (V_1 \cup V_2) \backslash \{1, 2\})) + x(\delta(V_1 \backslash \{1\}, V_2 \backslash \{2\})) \leq |\tilde{K}| + 1 \qquad (7)$$

Assume $(x, y) \in \mathcal{P}_{12}$. If $y_1 = y_2 = 1$, $x_{ij} \leq y_i, x_{ij} \leq y_j$ imply $x(\bigcup_{k \in \tilde{K}} \delta(V_k, (V_1 \cup V_2) \backslash \{1, 2\})) +$
$x(\delta(V_1) \backslash \{1\}, V_2 \backslash \{2\})) = 0$. In this case (7) is rewritten $x_{12} \leq 1$, which is valid for $\mathcal{P}_{12}$. Otherwise $x_{12} = 0$, and constraints (1) and $(2, l, \bigcup_{k \in \tilde{K}} V_k \cup V_1 \backslash \{1\} \cup V_2 \backslash \{2\}), l \in \tilde{K}$ imply (7). □

To prove that (6) induces a facet of $\mathcal{P}_{123}$, two technical lemmas are needed.

Lemma 3. *Let* $\alpha x \leq \beta y + \gamma$ *be a valid inequality inducing a proper face of* $\mathcal{P}_{123}$. *If there exist a pair of vertices* $i, j \in V \backslash W$ *such that all of the following three conditions hold:*

(i) if $h(i) = h(j)$ *then* $V_{h(i)} \backslash \{i, j\} \neq \emptyset$,
(ii) the ij*-restriction of* $\alpha x \leq \beta y + \gamma$ *is facet defining for* $\mathcal{P}_{123}^{ij}$,
(iii) there exist $|\delta(i) \cup \delta(j)| + 2$ *GST of the graph* G *containing at least one vertex of* $\{i, j\}$ *and whose incidence vectors are such that their restriction to* x_e *for all* $e \in \delta(i) \cup \delta(j)$ *and* y_i, y_j *are linearly independent,*

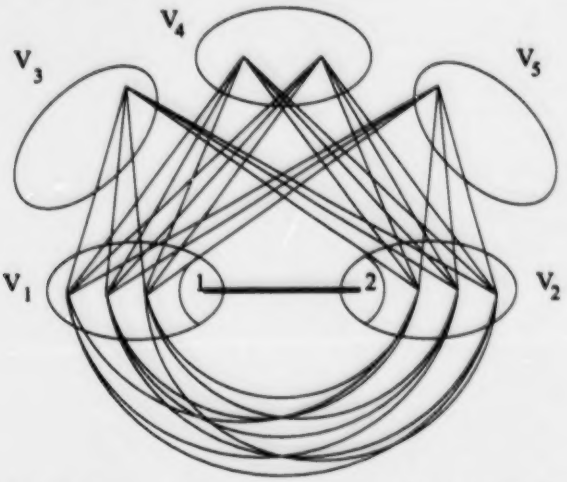

Fig. 1. Support graph of the hammock constraint for $r = 1, s = 2, h(r) = 1, h(s) = 2$ and $\tilde{K} = \{3, 4, 5\}$.

then $\alpha x \leq \beta y + \gamma$ *induces a facet of* $\mathcal{P}_{123}$.

Proof. Since $\mathcal{H}(\alpha, \beta, \gamma)$ is a proper face of $\mathcal{P}_{123}$ then $\dim(\mathcal{H}(\alpha, \beta, \gamma)) \leq |E| + |V| - |K| - 2$. Hypotheses (i), (ii) and Proposition 1 imply that $\dim(ij\text{-restriction}) = |E \backslash (\delta(i) \cup \delta(j))| + |V \backslash \{i, j\}| - |K| - 2 = |E| + |V| - |K| - |\delta(i) \cup \delta(j)| - 4$. Let $(X_{22} \quad Y_i \quad Y_j)$ be such that the columns of X_{22} represent the coordinates $x_e \in \delta(i) \cup \delta(j)$ and Y_i, Y_j correspond to the coordinates y_i, y_j of the extreme points of $\mathcal{H}(\alpha, \beta, \gamma)$, respectively, with $y_i + y_j \geq 1$. Hypothesis (iii) is equivalent to $\text{rank}(X_{22} \quad Y_i \quad Y_j) \geq |\delta(i) \cup \delta(j)| + 2$. Using the same argument as in Lemma 1, it implies $\dim(\mathcal{H}(\alpha, \beta, \gamma)) \geq |E| + |V| - |K| - 2$. Therefore $\dim(\mathcal{H}(\alpha, \beta, \gamma)) = |E| + |V| - |K| - 2$ and $\alpha x \leq \beta y + \gamma$ defines a facet of $\mathcal{P}_{123}$. □

Lemma 4. *Let* $V_1 = \{1, 3\}, V_2 = \{2, 4\}$, *and* $|V_k| = 1$ *for* $k \in \tilde{K} \subseteq K \backslash \{1, 2\}, |\tilde{K}| \geq 2$. *Then, there exist* $5 + 2|\tilde{K}|$ *incidence vectors* (x, y) *of GST satisfying constraint* (7) *relative to* $\tilde{K}$ *to equality and such that their restrictions to variables* x_e *for* $e \in (\delta(3) \cup \delta(4)) \cap \delta(\bigcup_{k \in \tilde{K}} V_k \cup V_1 \cup V_2)$ *and* y_i *for* $i \in \{3, 4\}$ *are linearly independent, and* $x_e = 0$ *for* $e \in (\delta(3) \cup \delta(4)) \backslash \delta(\bigcup_{k \in \tilde{K}} V_k \cup V_1 \cup V_2)$.

Proof. Let v_k be a vertex arbitrarily chosen in each V_k for $k = 1, \cdots, |K|$. The proof uses induction on $\rho = |\tilde{K}|$. First, if $|\tilde{K}| = 2$, let $\bigcup_{k \in \tilde{K}} V_k = \{v_3, v_4\}$. The following nine GST satisfy the requirements:

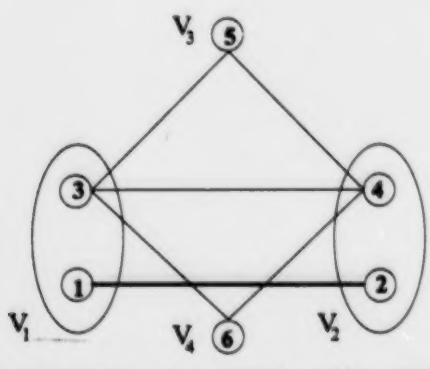

Fig. 2. Support graph of the hammock constraint for $r = 1, s = 2, h(r) = 1, h(s) = 2, V_1 = \{1, 3\}, V_2 = \{2, 4\}$, and $|V_k| = 1, k \in \tilde{K}, \tilde{K} = \{3, 4\}$.

$$\{(3,2)\} \cup \{(3,v_k) : k \in \tilde{K}\} \cup \{(v_3,v_k) : k \notin \tilde{K}\},$$
$$\{(3,4)\} \cup \{(3,v_k) : k \in \tilde{K}\} \cup \{(v_3,v_k) : k \notin \tilde{K}\},$$
$$\{(3,v_3)\} \cup \{(4,v_k) : k \in \tilde{K}\} \cup \{(v_3,v_k) : k \notin \tilde{K}\},$$
$$\{(3,v_4)\} \cup \{(4,v_k) : k \in \tilde{K}\} \cup \{(v_3,v_k) : k \notin \tilde{K}\},$$
$$\{(1,4)\}) \cup \{(4,v_k) : k \in \tilde{K}\} \cup \{(v_3,v_k) : k \notin \tilde{K}\},$$
$$\{(1,v_4)\} \cup \{(4,v_k) : k \in \tilde{K}\} \cup \{(v_3,v_k) : k \notin \tilde{K}\},$$
$$\{(3,4),(3,v_3),(4,v_4)\} \cup \{(v_3,v_k) : k \notin \tilde{K}\},$$
$$\{(3,4)\} \cup \{(4,v_k) : k \in \tilde{K}\} \cup \{(v_3,v_k) : k \notin \tilde{K}\},$$
$$\{(2,v_4)\} \cup \{(3,v_k) : k \in \tilde{K}\} \cup \{(v_3,v_k) : k \notin \tilde{K}\}.$$

Assume that the required vectors exist for all $\tilde{K}$ with $|\tilde{K}| \leq \rho$. Let $I \subseteq K\backslash\{1,2\}$, we must show that the claim still holds for $\tilde{K} = I \cup \{\tilde{k}\}$ with $|I| = \rho$, $\tilde{k} \in K\backslash(I \cup \{1,2\})$. We first transform the $5+2|I|$ vectors satisfying the conditions for I into vectors satisfying the conditions for $\tilde{K} = I \cup \{\tilde{k}\}$. If $y_3 = 1$, then we set $x_e = 1$ for $e = \{3, v_{\tilde{k}}\}$. Otherwise, we set $x_e = 1$ for $e = \{4, v_{\tilde{k}}\}$. The induction hypothesis implies that the restrictions of these incidence vectors to x_e for $e \in (\delta(3) \cup \delta(4)) \cap \delta(\bigcup_{k \in I \cup \{\tilde{k}\}} V_k \cup V_1 \cup V_2)$ and y_i for $i \in \{3,4\}$ are linearly independent. To conclude, we exhibit two linearly independent incidence vectors of GST such that their restriction to x_e for $e \in (\delta(3) \cup \delta(4)) \cap \delta(\bigcup_{k \in I \cup \{\tilde{k}\}} V_k \cup V_1 \cup V_2)$ and y_i for $i \in \{3,4\}$ are linearly independent from the previous $5+2|I|$ vectors and satisfy constraint (7) relative to $I \cup \{\tilde{k}\}$ to equality and $x_e = 0$ for $e \in (\delta(3) \cup \delta(4))\backslash\delta(\bigcup_{k \in I \cup \{\tilde{k}\}} V_k \cup V_1 \cup V_2)$. The following two GST satisfy the requirements:

$$\{(3,4),(4,v_{\tilde{k}})\} \cup \{(3,v_k) : k \in I\} \cup \{(v_{\tilde{k}},v_k) : k \notin I \cup \{1,2,\tilde{k}\}\} \quad \text{GST(1)},$$
$$\{(3,v_{\tilde{k}}),(4,v_{\tilde{k}})\}\} \cup \{(3,v_k) : k \in I\} \cup \{(v_{\tilde{k}},v_k) : k \notin I \cup \{1,2,\tilde{k}\}\} \quad \text{GST(2)}.$$

These trees clearly belong to the face induced by (7) relative to $\tilde{K} = I \cup \{\tilde{k}\}$. It remains to show that the restrictions of these two vectors form, together with the restrictions of the first $5+2|I|$ vectors, a set of linearly independent vectors. To conclude the proof, we substract a particular linear combination of the previous $5+2|I|$ restrictions from (i) and (ii). It then remains to show that the restriction to x_e for $e \in (\delta(3) \cup \delta(4)) \cap \delta(\bigcup_{k \in I \cup \{\tilde{k}\}} V_k \cup V_1 \cup V_2)$ and y_i for $i \in \{3,4\}$ of these two vectors have non-zero coordinates only for $e \in \{(3,v_{\tilde{k}}),(4,v_{\tilde{k}})\}$. For this purpose, consider the following GST:

$$\{(3,4)\} \cup \{(3,v_k) : k \in I \cup \{\tilde{k}\}\} \cup \{(v_{\tilde{k}},v_k) k \notin I \cup \{1,2,\tilde{k}\}\} \quad \text{GST(3)},$$
$$\{(3,v_m),(3,v_{\tilde{k}})\} \cup \{(4,v_k) : k \in I\} \cup \{(v_{\tilde{k}},v_k) k \notin I \cup \{1,2,\tilde{k}\}\}, m \in I \quad \text{GST}(4_m),$$
$$\{(3,4),(3,v_{\tilde{k}})\} \cup \{(4,v_k) : k \in I\} \cup \{(v_{\tilde{k}},v_k) k \notin I \cup \{1,2,\tilde{k}\}\}, \quad \text{GST(5)}.$$

Denote by g_l the incidence vector of GST(l). Then the combinations $g_1 - g_3$ and $g_2 + (|I|(g_5 - g_3) - \sum_{m \in I} g_{4_m})/(|I| + 1)$ satisfy the requirements. Indeed, the restrictions of these two incidence vectors to x_e for $e \in (\delta(3) \cup \delta(4)) \cap \delta(\bigcup_{k \in I \cup \{\tilde{k}\}} V_k \cup V_1 \cup V_2)$ and y_i for $i \in \{3,4\}$ have non-zero coordinates only for $x_e, e \in \{(3,v_{\tilde{k}}),(4,v_{\tilde{k}})\}$. More precisely, the coordinates $x_e, e \in \{(3,v_{\tilde{k}}),(4,v_{\tilde{k}})\}$ for these two combinations are respectively $(-1,1)$ and $(1/|\tilde{K}|,1)$ which are linearly independent. □

Proposition 7. *The Hammock constraints (7) are facet defining for $\mathcal{P}_{123}$, but not for $\mathcal{P}_{12}$.*

Proof. The proof is divided into two parts. First, we will show that if $|V_1| = |V_2| = 2$ and $|V_k| = 1$ for $k \in \tilde{K}, |\tilde{K}| \geq 2$, then the claim holds. Let v_k be a vertex arbitrarily chosen in each V_k for $k = 3, \cdots, |K|$.

Using Lemma 3 and Proposition 3(ii), we must exhibit $(|\delta(3) \cup \delta(4)| + 2)$ GST containing at least one vertex from $\{3,4\}$ and satisfying (7) to equality, and such that the restriction of their incidence vectors to y_3, y_4 and $x \in \delta(3) \cup \delta(4)$ are linearly independent. For every vertex $v \in V\backslash(\bigcup_{k \in \tilde{K}} V_k \cup V_1 \cup V_2)$, one can construct two GST $\{(j,v),(3,4)\} \cup \{(3,v_k) : k \in \tilde{K}\} \cup \{(v,v_k) : k \notin \tilde{K} \cup \{1,2,h(v)\}\}$ for $j = 3$ and 4 satisfying these requirements. This yields $|\delta(3) \cup \delta(4)| - 2|\tilde{K}| - 3$ of the required vectors. Lemma 4 gives the last needed vectors and this concludes the first part of the proof.

In the second part, we use induction on $\rho = |(\bigcup_{k \in \tilde{K}} V_k \cup V_1 \cup V_2)\backslash\{1,2\}| - |\tilde{K}| - 2$. The first part of the proof shows

that the claim is true when $\rho = 0$. Assume now that the claim holds for $\rho \leq \bar{\rho}$ and consider any GST polytope with $\rho = \bar{\rho} + 1$. Then there exists a vertex $v \in (\bigcup_{k \in \tilde{K}} V_k \cup V_1 \cup V_2) \backslash (W \cup \{1,2\})$. This induction assumption together with Lemma 2, implies that it suffices to exhibit $|\delta(v)| + 1$ vectors satisfying the conditions of the lemma. There are two cases to consider.

Case 1. If $v \in \bigcup_{k \in \tilde{K}} V_k$, assume that $h(v) = 3$. The $|\delta(v)| + 1$ incidence vectors associated with the following trees meet the requirements:

$$\begin{array}{ll} \{(v,p),(1,2),(2,p)\} \cup \{(p,v_k) : k \in K \backslash \{1,2,3,h(p)\}\} & \text{for } p \in V \backslash \bigcup_{j=1}^{3} V_j, \\ \{(1,2),(v,p)\} \cup \{(p,v_k) : k \in K \backslash \{1,2,3\}\} & \text{for } p \in \{1,2\}, \\ \{(3,4),(v,p)\} \cup \{(p,v_k) : k \in K \backslash \{1,2,3\}\} & \text{for } p \in \{3,4\}. \end{array}$$

Since $|\tilde{K}| \geq 2$, there exists q with $h(q) \in \tilde{K}, h(q) \neq 3$ and the last vector can be $\{(1,2),(v,2),(v,q)\} \cup \{(q,v_k) : k \in K \backslash \{1,2,3,h(q)\}\}$.

Case 2. If $v \in (V_1 \cup V_2) \backslash (W \cup \{1,2\})$, assume that $h(v) = 1$. The $|\delta(v)| + 1$ incidence vectors associated with the following trees meet the requirements:

$$\begin{array}{ll} \{(v,p),(v,4)\} \cup \{(4,v_k) : k \in K \backslash \{1,2,h(p)\}\} & \text{for } p \in V \backslash \bigcup_{k \in \tilde{K} \cup \{1,2\}} V_k, \\ \{(v,p),(4,p)\} \cup \{(4,v_k) : k \in K \backslash \{1,2,h(p)\}\} & \text{for } p \in \bigcup_{k \in \tilde{K}} V_k, \\ \{(v,p)\} \cup \{(v,v_k) : k \in \tilde{K}\} \cup \{(p,v_k) : k \notin \tilde{K} \cup \{1,2\}\} & \text{for } p \in V_2. \end{array}$$

Let $\tilde{k} \in \tilde{K}$, the last vector can be associated with $\{(v,v_k) : k \in \tilde{K}\} \cup \{(2,v_k) : k \in (K \backslash (\tilde{K} \cup \{1,2\})) \cup \{\tilde{k}\}\}$.

To conclude the proof, we observe that all vectors of $\mathcal{P}_{12}$ on the face induced by the hammock constraint belong to the hyperplane defined by
$y(V_1) + y(V_2) = 2$. □

4. Polytope projections in the x-space.

In this section we investigate relationships between polytopes $\mathcal{P}_{12}$ and $\mathcal{P}_1$ defined in the (x,y)-space and their projection in x-space.

Definition 2. *The polytope in the x-space associated with $\mathcal{P}_{12}, \mathcal{P}_1$ is defined as $\mathcal{P}_{12}^x = proj_x(\mathcal{P}_{12}), \mathcal{P}_1^x = proj_x(\mathcal{P}_{12})$, where $proj_x$ denotes the orthogonal projection in the x-space.*

Obviously, $\mathcal{P}_{12}^x \subset \mathcal{P}_1^x$, and $\mathcal{P}_{12}^x, \mathcal{P}_1^x$ are full-dimensional.

Theorem 1. *If the following two conditions hold:*

(i) the inequality $\pi x \leq \pi_0$ is facet defining for $\mathcal{P}_{12}^x$,
(ii) for every cluster V_k, there exists a solution $\bar{x}^k \in \mathcal{P}_{12}^x \cap \{0,1\}^{|E|}$ such that $\pi \bar{x}^k = \pi_0$ and $\bar{x}_e^k = 0$ for all $e \in \delta(V_k)$,

then

(iii) the inequality $\pi x \leq \pi_0$ is facet defining for $\mathcal{P}_{12}$ and,
(iv) if in addition $\pi x \leq \pi_0$ is valid for $\mathcal{P}_1$, it then defines a facet of $\mathcal{P}_1$.

Proof. We will use the indirect method to prove that $\pi x \leq \pi_0$ is facet inducing for $\mathcal{P}_{12}$. This is done by showing that any other inequality of $\mathcal{P}_{12}$

$$\sum_{e \in E} \alpha_e x_e + \sum_{i \in V} \beta_i y_i \leq \alpha_0$$

such that $\{(x,y) \in \mathcal{P}_{12}, \pi x = \pi_0\} \subset \{(x,y) \in \mathcal{P}_{12}, \alpha x + \beta y = \alpha_0\}$ must be a non-negative multiple of $\pi x \leq \pi_0$. Since $\pi x = \pi_0$ contains for every $k \in K$ a solution $\bar{x}^k \in \mathcal{P}_{12}^x \cap \{0,1\}^{|E|}$ such that $\bar{x}_e^k = 0$ for all $e \in \delta(V_k)$, we

can extend this solution into several solutions $(\tilde{x}^k, \tilde{y}) \in \mathcal{P}_{12} \cap \{0,1\}^{|E||V|}$. Using the solution defined as follows: $\tilde{y}_i = 1$ if there exists $e \in \delta(i)$ such that $\tilde{x}^k_e = 1$, $\tilde{y}_i = 0$ otherwise,

$$\sum_{e \in E \setminus \delta(V_k), \tilde{x}^k_e = 1} \alpha_e + \sum_{p \in \{i,j\} = e, \tilde{x}^k_e = 1} \beta_p = \alpha_0.$$

For every vertex $v \in V_k$ we can construct the following solution: $\tilde{y}_i = 1$ if there exists $e \in \delta(i)$ such that $\tilde{x}^k_e = 1, \tilde{y}_v = 1$, and $\tilde{y}_i = 0$ otherwise,

$$\sum_{e \in E \setminus \delta(V_k), \tilde{x}^k_e = 1} \alpha_e + \sum_{p \in \{i,j\} = e, \tilde{x}^k_e = 1} \beta_p + \beta_v = \alpha_0.$$

These two constraints imply that $\beta_v = 0$ for all $v \in V_k$. Since we assumed that there exists a solution $\tilde{x}^k$ for every $k \in K$, we can conclude that $\beta_v = 0$ for all $v \in V$.
If $\{(x,y) \in \mathcal{P}_{12}, \pi x = \pi_0\} \subset \{(x,y) \in \mathcal{P}_{12}, \alpha x = \alpha_0\}$, then $\{x \in \mathcal{P}^x_{12}, \pi x = \pi_0\} \subset \{x \in \mathcal{P}^x_{12}, \alpha x = \alpha_0\}$. Since we assumed that $\pi x \leq \pi_0$ is facet defining for $\mathcal{P}^x_{12}$ this implies that (α, α_0) is proportional to (π, π_0). □

Note that if $\pi x \leq \pi_0$ is facet defining for $\mathcal{P}^x_{12}$ and is valid for $\mathcal{P}^x_1$, it then defines a facet of $\mathcal{P}^x_1$. The same holds for $\mathcal{P}_{12}$ and $\mathcal{P}_1$.

Theorem 2. *If the following two conditions hold:*

(i) the inequality $\pi x \leq \pi_0$ is facet defining for $\mathcal{P}^x_1$,
(ii) for every cluster V_k, there exists a solution $\tilde{x}^k \in \mathcal{P}^x_1 \cap \{0,1\}^{|E|}$ such that $\pi \tilde{x}^k = \pi_0$ and $\tilde{x}^k_e = 0$ for all $e \in \delta(V_k)$,

then $\pi x \leq \pi_0$ is facet defining for $\mathcal{P}_1$.

Proof. The proof of Theorem 2 is along the same lines as that of Theorem 1.

We now investigate two new families of valid inequalities for $\mathcal{P}^x_{12}$. Using Theorem 1, we prove that they are facet inducing for $\mathcal{P}^x_1, \mathcal{P}_{12}$ and $\mathcal{P}_1$.

Proposition 8. *(Odd Cycle inequalities). Let $V_1, \cdots, V_m$, where m is odd, and define partitions $\{V^1_k, V^2_k\}$ clusters for $k = 1, \cdots, m$. Then the inequality*

$$\sum_{k=1}^{m} x(\delta(V^1_k, V^2_{(k \bmod m)+1})) \leq \lfloor m/2 \rfloor \tag{8}$$

is valid for $\mathcal{P}_1$ (and thus for $\mathcal{P}_{12}$ and $\mathcal{P}_{123}$).

Proof. The validity of (8) follows from the fact that a feasible GS solution contains at most one vertex per cluster. □

Proposition 9. *If the subclusters V^l_k of Proposition 8 are all non-empty then the inequality (8) is facet defining for $\mathcal{P}^x_{12}$, $\mathcal{P}^x_1$, $\mathcal{P}_{12}$ and $\mathcal{P}_1$.*

Proof. The proof contains three steps. First, we show using the indirect method that if the graph G contains only the m clusters and only the edges whose x variable appears in (8), then (8) is facet defining for $\mathcal{P}^x_{12}$. Second, we show that when G is complete $|K|$-partite, the coefficients of the remaining edges are equal to zero in an optimal lifting. In a third step, we check that the conditions of Theorem 1 are satisfied. For simplicity, let $\delta_k = \delta(V^1_k, V^2_{(k \bmod m)+1})$ $(k = 1, \cdots, m)$, and $\Delta = \bigcup_{k=1}^{m} \delta_k$.
In the first step, we show that any other inequality $\sum_{e \in E} \alpha_e x_e \leq \alpha_0$ such that $\{x \in \mathcal{P}^x_{12} : x(\Delta) = \lfloor m/2 \rfloor\} \subseteq \{x \in \mathcal{P}^x_{12} : \alpha x = \alpha_0\}$ must be a non-negative multiple of $x(\Delta) \leq \lfloor m/2 \rfloor$. We first prove that all coefficients α_e associated with the edges $e \in \delta_1 \cup \delta_2$ are equal. For this purpose define a set F of edges satisfying $|F| = \lfloor m/2 \rfloor - 1$ and $|F \cap \delta_k| = 1$ for $k = 4, 6, \cdots, m-1$ and consider the vectors $u^E_e + \sum_{f \in F} u^E_f$, for $e \in \delta_1 \cup \delta_2$. These vectors all belong to the face of $\mathcal{P}^x_{12}$ induced by (8). Hence they all satisfy $\alpha x = \alpha_0$, which implies that all α_e coefficients associated with the edges e in $\delta_1 \cup \delta_2$ are equal. The same reasoning applies *mutatis mutandis* if e is initially

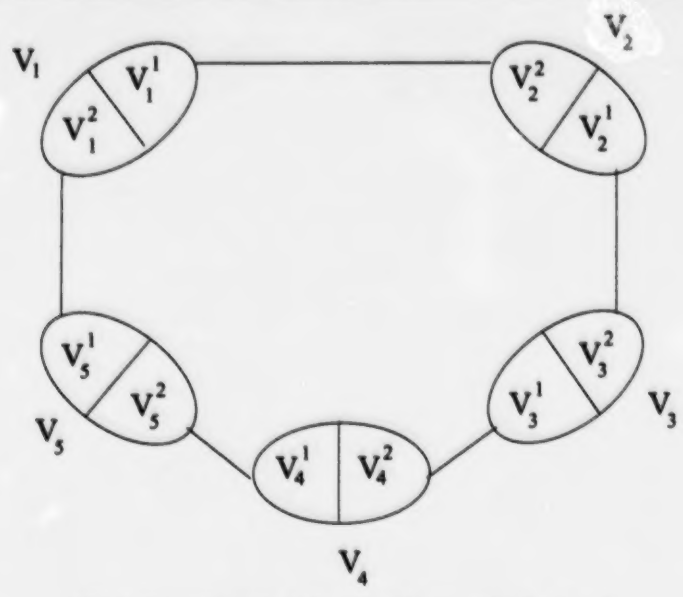

Fig. 3. Support graph of the odd cycle constraint for $m = 5$.

chosen in $\delta_k \cup \delta_{(k \bmod m)+1}$ for $k = 2, \cdots, m$. It follows that $\alpha_e = \alpha$ for all $e \in \Delta$ and $\alpha_0 = \alpha\lfloor m/2 \rfloor$. For the second step, we observe that for any edge $e \in E \backslash \Delta$, it is possible to select $\lfloor m/2 \rfloor$ edges in Δ which together with e form a GAS. Hence the coefficient α_e must be equal to zero in an optimal lifting. This show that (8) is facet defining for $\mathcal{P}_{12}^s$. The third step of the proof follows directly from the fact that for any given $k \in K$, there exists a GAS solution containing exactly $\lfloor m/2 \rfloor$ edges, all belonging to Δ and thus having an empty intersection with $\delta(V_k)$. □

Proposition 10. *(Odd clique-matching inequalities). Let $V_1, \cdots, V_m$ be m clusters where m is odd, and define partitions $\{V_k^l\}$ for $k = 1, \cdots, m$ and $l = 1, \cdots, m-1$. Then*

$$\sum_{k=1}^{m} \sum_{l=k}^{m-1} x(\delta(V_k^l, V_{l+1}^k)) \leq \lfloor m/2 \rfloor \tag{9}$$

is a valid inequality for $\mathcal{P}_1$ (and thus for $\mathcal{P}_{12}$ and $\mathcal{P}_{123}$).

Proof. The validity of (9) follows from the fact that a feasible GS solution contains at most one vertex per cluster. . □

Proposition 11. *If the subclusters V_k^l of Proposition 10 are all non-empty, then the inequality (9) is facet defining for $\mathcal{P}_{12}^s, \mathcal{P}_1^s, \mathcal{P}_{12}$, and $\mathcal{P}_1$.*

Proof. Let $\Delta = \bigcup_{k=1}^{m} \bigcup_{l=k}^{m-1} \delta(V_k^l, V_{l+1}^k)$. The proof works along the same lines of that of Proposition 9. Initially, we show that all coefficient α_e associated with the edges $e \in \delta(V_1^1, V_2^1) \cup \delta(V_2^2, V_3^2) \cup \delta(V_3^2, V_3^1)$ are equal in $\sum_{e \in \Delta} \alpha_e x_e \leq \alpha_0$, such that $\{x \in \mathcal{P}_{12}^s : (9) \text{ holds}\} \subseteq \{x \in \mathcal{P}_{12}^s : \alpha x = \alpha_0\}$. Indeed for any such e, there exists a set F of $\lfloor m/2 \rfloor - 1$ edges in $\Delta \backslash (\delta(V_1^1, V_2^1) \cup \delta(V_2^2, V_3^2) \cup \delta(V_3^2, V_3^1))$ which together with e constitute a GAS. This concludes the first step of the argument. Steps 2 and 3 are similar to those of Proposition 9. □

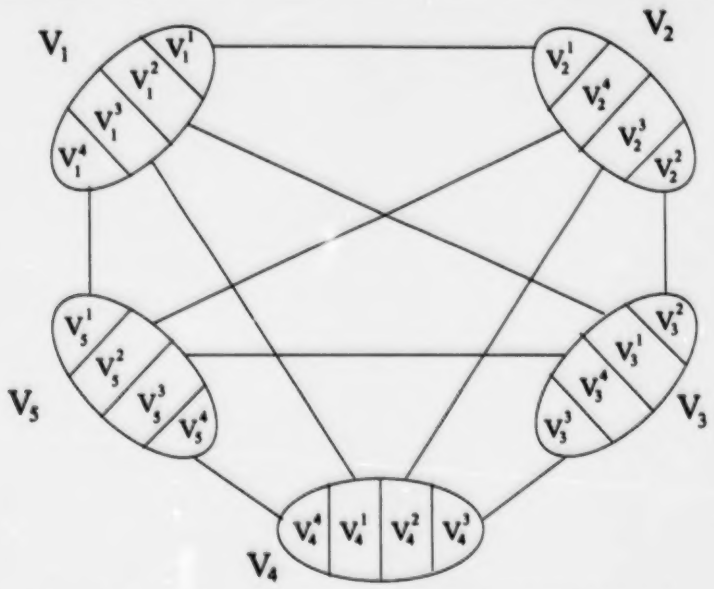

Fig. 4. Support graph of the odd clique-matching constraint for $m = 5$.

5. Summary and conclusion

We have analyzed the polyhedral structure of the polytopes $\mathcal{P}_{123}, \mathcal{P}_{12}$ and $\mathcal{P}_1$ associated with the GST, the GAS and the GS respectively. These polytopes satisfy the relationship $\mathcal{P}_{123} \subseteq \mathcal{P}_{12} \subseteq \mathcal{P}_1$. Several classes of constraints have been shown to be valid and sometimes facet defining for these polytopes and for the two projections $\mathcal{P}_{12}^x$ and $\mathcal{P}_1^x$. Three families of constraints, the hammock inequalities, the odd cycle inequalities, and the odd clique-matching inequalities are new. Table 1 summarizes the main findings of this study and shows that two polyhedral properties are still unresolved.

Type of constraint	Constraint	Facets of the polytopes?				
		$\mathcal{P}_{123}$	$\mathcal{P}_{12}$	$\mathcal{P}_1$	$\mathcal{P}_{12}^x$	$\mathcal{P}_1^x$
Bound	$x_e \geq 0$	Yes	Yes	Yes	Yes	Yes
	$y_i \geq 0$	No	No	No	n.a.	n.a.
	$y(V_k) \leq 1$	No	Yes	Yes	n.a.	n.a.
Subtour Elimination	$x_e \leq 1$	iff $e \in E(W)$	Yes	Yes	Yes	Yes
	$x(\delta(v)) \geq y_v$	Yes	No	No	n.a.	n.a.
	$x(\delta(v, V_k)) \leq y_v$	iff $v \notin W \cup V_k$	iff $v \notin V_k$	iff $v \notin V_k$	n.a.	n.a.
	$(2, k, S), V_k \subset S, \eta(S) \geq 3$	iff $\eta(V \setminus S) \geq 2$	Yes	No	n.a.	n.a.
Hammock	(6)	Yes	No	No	n.a.	n.a.
Odd cycle	(8)	Open	Yes	Yes	Yes	Yes
Odd clique-matching	(9)	Open	Yes	Yes	Yes	Yes

Table 1. Summary of the results.

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